Exercise Set Solutions #4

"Discrete Mathematics" (2025)

E1. In how many ways can you write $3, 7$, and 12 as a sum of three numbers chosen from the sets $\{1, 2, 3, 5, 8\}$, $\{2, 3, 5, 7\}$, and $\{0, 2, 4\}$ respectively?

> ***Solution:*** Consider the following product.
>
> $$\left(x + x^2 + x^3 + x^5 + x^8\right)\left(x^2 + x^3 + x^5 + x^7\right)\left(1 + x^2 + x^4\right) = \\ x^{19} + 2x^{17} + x^{16} + 3x^{15} + 4x^{14} + 3x^{13} + 7x^{12} + 4x^{11} \\ +8x^{10} + 5x^9 + 7x^8 + 5x^7 + 4x^6 + 3x^5 + 2x^4 + x^3$$
>
> The coefficients of x^3, x^7, x^12 are $1, 5, 7$ respectively. These are precisely the answer.

E2. Find the sequence generated by the generating function:

(a) $\frac{x^3}{(1+x)^2}$.

(b) $\frac{1+x+x^2}{(1-x)^2}$.

> ***Solution:*** (a) The idea is to manipulate an expression that we know until we obtain the wanted result. While doing so we have to keep track of both sides, i.e. the closed expression of the geometric series and the sequence that corresponds to it. In the lecture we have shown that for $|x| < 1$, we have $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$. By taking the derivative on both sides we get $\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^k$. Next we substitute x by $-x$ and obtain
>
> $$\frac{1}{(1+x)^2} = \sum_{k=0}^{\infty}(k+1)(-x)^k$$
>
> The corresponding sequence is now $1, -2, 3, -4, \ldots$ By multiplying both sides with x^3, we obtain
>
> $$\frac{x^3}{(1+x)^2} = -\sum_{k=0}^{\infty}(k+1)(-x)^{k+3} = -x^3 + 2x^4 - 3x^5 + 4x^6 - \ldots$$
>
> Therefore, the sequence generated by the generating function $\frac{x^3}{(1+x)^2}$ is $(a_0, a_1, a_2, \ldots)$, where
>
> $$a_k = \begin{cases} 0 & \text{if } k \in \{0, 1, 2\} \\ (-1)^{k+1}(k-2) & \text{otherwise} \end{cases}$$
>
> (b) After differentiating the series of $\frac{1}{1-x}$ for $|x| < 1$, we have
>
> $$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^k$$

and the corresponding sequence is $a = (1, 2, 3, 4, \ldots)$. If we multiply both sides with x, we shift the sequence by one position.

$$\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^{k+1}$$

and the corresponding sequence is $b = (0, 1, 2, 3, 4, 5, \ldots)$. Analogously, by multiplying again by x we get

$$\frac{x^2}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^{k+2}$$

with the sequence $c = (0, 0, 1, 2, 3, 4, 5, \ldots)$. Hence, to obtaint the sequence for $\frac{1+x+x^2}{(1-x)^2}$ we have to take the sum of the sequences a, b and c. Since $a + b + c = (1, 3, 6, 9, 12, 15, \ldots)$ this is the sequence we are looking for.

E3. Determine the generating function of the following sequences and write them as a closed expression.

(a) $(a_0, a_1, a_2, a_3, a_4, \ldots) = (1, 3, 5, 7, 9, \ldots)$;

(b) $(a_0, a_1, a_2, a_3, a_4, \ldots) = (1, 0, 1, 0, 1, \ldots)$;

(c) $a_n = n^2 + 1$ for each $n \in \mathbb{N}$;

Solution: Let $a(x)$ be the generating function associated with $\{a_n\}_n$ in each of the items below.

(a) $a(x) = 1 + 3x + 5x^2 + 7x^3 + \ldots$, then $xa(x) = x + 3x^2 + 5x^3 + 7x^4 + \ldots$. Therefore $(1-x)a(x) = 1 + 2x + 2x^2 + 2x^3 + \ldots$. We know that $2x\left(1 + x + x^2 + x^3 + \ldots\right) = \frac{2x}{(1-x)}$. Hence, $(1-x)a(x) = 1 + \frac{2x}{(1-x)}$ and then we get

$$a(x) = \frac{1}{1-x} + \frac{2x}{(1-x)^2} = \frac{1+x}{(1-x)^2}$$

(b) $a(x) = 1 + x^2 + x^4 \cdots$. Since we know that $1 + y + y^2 + y^3 + \cdots = \frac{1}{1-y}$, taking $y = x^2$, we have

$$a(x) = \frac{1}{1-x^2}.$$

(c) Let $A(x)$ be the generating function associated to $(a_n)_n$. Notice that

$$\sum_{n=0}((n+1)^2+1)x^n = \sum_{n=0}^{\infty}(n^2+2n+2)x^2 = A(x)+2\sum_{n=0}^{\infty}(n+1)x^n) = A(x)+2\frac{d}{dx}(\sum_{n=0}^{\infty}x^{n+1}).$$

Observe that

$$\sum_{n=0}^{\infty}x^{n+1} = \sum_{n=1}^{\infty}x^n = \frac{1}{1-x} - 1.$$

Thus

$$\sum_{n=0}((n+1)^2+1)x^n = A(x) + \frac{2}{(1-x)^2}.$$

On the other hand, notice that

$$\sum_{n=0}((n+1)^2+1)x^n = \frac{1}{x}\sum_{n=1}^{\infty}(n^2+1)x^n = \frac{1}{x}(A(x)-1).$$

Thus, combining this two equations we get

$$(A(x)-1) = x\cdot(A(x)+\frac{2}{(1-x)^2})$$

which implies

$$A(x) = \frac{1}{(1-x)} + \frac{2x}{(1-x)^3}.$$

E4. Prove the following using generating polynomials.

$$\sum_{k=q}^{n}\binom{n}{k}\binom{k}{q} = 2^{n-q}\binom{n}{q}.$$

Solution: Consider the polynomial $(2+x)^n$. What is the coefficient of x^q in this?

$$(2+x)^n = \sum_{q=0}^{n} x^q 2^{n-q}\binom{n}{q}$$

This gives us the RHS. For the LHS we note that there is no power of 2 anymore. We therefore rewrite 2 as $1+1$ and obtain

$$\begin{aligned}(2+x)^n &= (1+(1+x))^n\\ &= \sum_{k=0}^{n}(1+x)^k\binom{n}{k}\\ &= \sum_{k=0}^{n}\sum_{q=0}^{k} x^q\binom{k}{q}\binom{n}{k}\\ &= \sum_{q=0}^{k}\sum_{k=q}^{n} x^q\binom{k}{q}\binom{n}{k}\end{aligned}$$

E5. Let $\{a_n\}_{n\geq 0}$ denote the sequence of non-negative integers given by $a_0 = 1$ and $a_{n+1} = 2a_n + n$.

(a) Let

$$A(x) = \sum_{n\geq 0} a_n x^n$$

be the generating function associated with $\{a_n\}_{n\geq 0}$. Find $A(x)$.

(b) Give an explicit expression for a_n.

Solution: First, we notice that $x^{n+1}a_{n+1} = 2x^{n+1}a_n + x^{n+1}n$ for each $n \in \mathbb{N}$. Adding over $\mathbb{N}_0$

we get

$$\sum_{n=0}^{\infty} x^{n+1}a_{n+1} = \sum_{n=0}^{\infty} 2x^{n+1}a_n + x^{n+1}n.$$

The left-hand side is equal to $A(x) - a_0 = A(x) - 1$. Meanwhile, the right-hand side is equal to

$$2x\sum_{n=0}^{\infty} x^n a_n + x^2 \sum_{n=1}^{\infty} nx^{n-1} = 2xA(x) + x^2 \frac{d}{dx}\sum_{n=1}^{\infty} x^n = 2xA(x) + x^2 \frac{d}{dx}\frac{1}{1-x} = 2xA(x) + \frac{x^2}{(1-x)^2}.$$

In consequence

$$A(x) = \frac{1}{1-2x} + \frac{x^2}{(1-2x)(1-x)^2}.$$

For finding $(a_n)_n$ explicitly, we define $b_n = a_n + n$. Notice that $b_0 = 1$ and $b_{n+1} = 2b_n + 1$, which translates into

$$B(x) - 1 = 2xB(x) + x\sum_{n=0}^{\infty} x^n$$

where $B(x) = \sum_{n=0}^{\infty} b_n x^n$. Thus

$$B(x) = \frac{1}{1-2x} + \frac{x}{1-2x}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 2^n x^n + \left(\sum_{n=1}^{\infty} 2^{n-1}x^n\right)\left(\sum_{n=0}^{\infty} x^n\right).$$

Hence

$$b_n = 2^n + \sum_{k=1}^{n} 2^{k-1} = 2^n + 2^n - 1.$$

Thus

$$a_n = -n + 2^{n+1} - 1.$$

E6. Suppose $n, k_1, k_2, \ldots, k_r \in \mathbb{Z}^{\geq 0}$ such that $\sum_{i=1}^{r} k_r = n$. We use the following notation for multinomials:

$$\binom{n}{k_1, k_2, k_3, \ldots, k_r} = \frac{n!}{k_1!k_2!\ldots k_r!}.$$

(a) Prove that

$$\binom{n}{k_1, k_2, \ldots, k_r} = \binom{n-1}{k_1 - 1, k_2, \ldots, k_r} + \binom{n-1}{k_1, k_2 - 1, k_3, \ldots, k_r} + \cdots \binom{n-1}{k_1, k_2, \ldots, k_r - 1}.$$

Compare this with Proposition 1.11 from the lecture about binomial coefficients.

Solution: The left hand is the coefficient of $x_1^{k_1}x_2^{k_2}\ldots x_r^{k_r}$ in $(x_1 + x_2 + \cdots + x_r)^n$. The right hand side is the same coefficient in $(x_1 + x_2 + \cdots + x_r)(x_1 + x_2 + \cdots + x_r)^{n-1}$.
Another way to see this is that the left side can be understood as the number of ways to arrange a sequence of the letters $a_1, a_2, \ldots, a_r$ where a_i appears exactly k_i times. But for each i the number of such sequences that begin with an a_i is exactly the i th term on the right.

(b) Show that

$$\sum_{k_1+k_2+\ldots+k_m=n} \binom{n}{k_1, k_2, \ldots k_m} = m^n$$

Solution: This follows by expanding $(x_1 + x_2 + \cdots + x_m)^n$ in coefficients and then letting $x_1 = x_2 = \cdots = 1$.